Duality and fields redefinition in three dimensions

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Abstract

We analyze local fields redefinition and duality for gauge field theories in three dimensions. We find that both Maxwell-Chern-Simons and the Self-Dual models admits the same fields redefinition. Maxwell-Proca action and its dual also share this property. We show explicitly that a gauge-fixing term has no influence on duality and fields redefinition.

1 Introduction

In this paper we investigate the relationship between duality and fields redefinition, in a sense introduced in [1] and [2], for Maxwell-Chern-Simons and Maxwell-Proca models in three space-time dimensions.

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It is widely known that a sufficient condition for duality is the existence of a global symmetry in the original action and one finds the dual model "gauging" this symmetry [3]. The original model is now seen as a gauge-fixed version of its dual model. One secure prescription for achieving that is "gauge embedding" procedure [4].

This formalism can be used to derive the well-known duality between self-dual model (SD) and Maxwell-Chern-Simons action (MCS) [5]. Some questions are natural consequences of this fact. The first one: this duality implies that both models will have the same field redefinition? The answer is yes, and we prove. A second question is if this property holds to some other dual gauge models. To provide an example, we deal with Maxwell-Proca action (MP) and its dual to show that they indeed have the same field redefinition. A more rigorous treatment about this apparent connection between duality and fields redefinition is given in [6].

This paper is organized as follows. In Section 2 we prove that SD and MCS models have the same fields redefinition. Section 3 is devoted to establish the same fact MP and its dual. In appendix A we detail the rules for functional calculus with differential forms.

We use the following conventions: the metric adopted is $(-++)$. We omit the wedge product symbol $\wedge$ and use the inner product of two p-forms $(\omega_p, \eta_p) = \int \omega_p \wedge^* \eta_p$. $*$ denotes the usual duality Hodge operator.

2 Field Redefinitions on Maxwell-Chern-Simons (MCS) and Self-Dual (SD) Models

Twenty years ago, Deser and Jackiw [5] established duality between (SD) action

$$S_{SD} = \int \left(\frac{m^2}{2} A^* A + \frac{1}{2} m A dA \right) = \frac{m^2}{2} (A, A) - \frac{m}{2} (A,^* dA), \qquad (2.1)$$

and topologically massive MCS action

$$S_{MCS} = \int \frac{1}{2} dA^* dA - \frac{m}{2} A dA = \frac{1}{2} (dA, dA) + \frac{m}{2} (A,^* dA), \qquad (2.2)$$

following the clue that both models involve a massive vectorial field in three dimensions and violate parity. They derive the duality from a master action. Each model is obtained combining equations of motion and this master action. This can be also achieved using gauge embedding formalism [4].

In [1] and [2] Lemes *et al* showed that Chern-Simons term can be seen as a generator for MCS model, through a field redefinition. We set the redefinition for

$$\frac{1}{2}(dA, dA) + \frac{m}{2}(A, *dA) = \frac{m}{2}\left(\hat{A}, *d\hat{A}\right),$$
(2.3)

where

$$\hat{A} = A + \sum_{i=1}^{\infty} \frac{A_i}{m^i}.$$
(2.4)

It was shown in [7] that

$$\delta A_i = 0,$$
(2.5a)
$$d^\dagger A_i = 0,$$
(2.5b)

and that A_i depends linearly on A in the following way

$$A_i = \alpha_i (*d)^i A.$$
(2.6)

We can also look at field redefinition (2.4) as having an operatorial nature[1],

$$\hat{A} = \mathcal{O}A,$$
(2.7)

where $\mathcal{O}$ is an operator defined by

$$\mathcal{O} = \left(1 + \sum_{j=1}^{\infty} \frac{\alpha_j (*d)^j}{m^j}\right).$$
(2.8)

The operator $\mathcal{O}$ has the following properties

$$(\mathcal{O}A, B) = (A, \mathcal{O}B),$$
(2.9a)
$$[\mathcal{O}, *d] = 0,$$
(2.9b)

for any one-forms A and B. So, we can write the field redefinition for the MCS model in the form

$$\frac{1}{2}(dA, dA) + \frac{m}{2}(A, *dA) = \frac{m}{2}(\mathcal{O}A, *d\mathcal{O}A).$$
(2.10)

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[1]Let us comment about a functional formulation with this field redefinition. Since this redefinition is linear, the functional measure gains a factor which does not depend on the field and hence is factorable from the functional.

Taking the functional derivative with respect to A on both sides of this equation we obtain

$$\left(\frac{*d}{m} - 1 + \mathcal{O}^2\right) * dA = 0. \tag{2.11}$$

Since the operator $\mathcal{O}$ depends only on $*d$, we must have

$$\mathcal{O}^2 = 1 - \frac{*d}{m}, \tag{2.12}$$

and consequently

$$\mathcal{O}^2 A = A - \frac{*d}{m} A. \tag{2.13}$$

Integrating with respect to A we get

$$\frac{1}{2} m^2 (A, A) - \frac{1}{2} m \left(A, *dA\right) = \frac{1}{2} m^2 \left(\hat{A}, \hat{A}\right). \tag{2.14}$$

So the fields redefinitions to MCS and its dual SD model are the same and SD model can be reset in a pure mass-like term. From (2.12) we can obtain the form of the operator $\mathcal{O}$ by expansion in power series in $*d/m$

$$\mathcal{O} = 1 - \sum_{j=1}^{\infty} \frac{(2j)!}{2^{2j} (2j-1) (j!)^2} \left(\frac{*d}{m}\right)^{j}. \tag{2.15}$$

Comparing (2.7) and (2.15), we find out a formula for the coefficients $\alpha_j\prime s$

$$\alpha_j = -\frac{(2j)!}{2^{2j} (2j-1) (j!)^2}, \tag{2.16}$$

that furnishes the same coefficients found in [8].

We can directly invert the field redefinition

$$A = \mathcal{O}^{-1} \hat{A}, \tag{2.17}$$

with

$$\mathcal{O}^{-1} = \left(1 - \frac{*d}{m}\right)^{-1/2} = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k} (k!)^2} \left(\frac{*d}{m}\right)^{k}, \tag{2.18}$$

that are also in complete agreement with that ones [8].

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The series expansion

$$(1+x)^{1/2} = \sum_{j=1}^{\infty} (-1)^j \alpha_j x^j, \qquad (2.19)$$

converges only for $|x| < 1$ and $x = 1$, but this is not a problem since one can easily extend the convergence for another intervals by analytic continuation.

We can change our way of thinking and consider as (2.11) had been obtained from (2.13) after application of $*d$. Thus redefined MCS is generated by redefined SD model, both presenting the same field redefinition. One can reasonably argue what would one get applying $*d$ twice on (2.13) and integrating on field A. We obtain a new gauge invariant model

$$-\frac{1}{2}(dA, d*dA) + \frac{m}{2}(dA, dA) = \frac{m}{2}\left(d\hat{A}, d\hat{A}\right). \qquad (2.20)$$

By construction the field redefinition of (2.20) is exactly that for SD and MCS models. In fact, these models belongs to a same equivalence class [6].

An interesting point is that adding a gauge-fixing term to SD model have no influence on fields redefinition since that it does not spoil duality. Consider the Self-Dual model plus a Landau gauge fixing term

$$S^{(0)} = \int \left(\frac{m^2}{2}A_\mu A^\mu + \frac{\alpha}{2}(\partial_\mu A^\mu)^2 - \frac{m}{2}\epsilon_{\mu\nu\rho}A^\rho\partial^\mu A^\nu\right) d^3x. \qquad (2.21)$$

Following the gauge embedding prescription [4], the first iterative action is

$$S^{(1)} = S^{(0)} - \int \left(m^2 A_\mu - \alpha\partial_\mu(\partial A) - m\epsilon_{\mu\nu\rho}\partial^\nu A^\rho\right) B^\mu d^3x, \qquad (2.22)$$

and it follows that

$$\delta S^{(1)} = -\delta \int \frac{m^2}{2}B_\mu B^\mu d^3x - \delta \int \frac{\alpha}{2}(\partial_\mu B^\mu)^2 d^3x. \qquad (2.23)$$

Then, the invariant action is

$$S^{(2)} = S^{(1)} + \int \left(\frac{m^2}{2}B_\mu B^\mu + \frac{\alpha}{2}(\partial_\mu B^\mu)^2\right) d^3x. \qquad (2.24)$$

The equation of motion for B_μ shows that $\partial_\mu B^\mu = \partial_\mu A^\mu$ and $m^2 B_\mu = m^2 A_\mu - m\epsilon_{\mu\nu\rho}\partial^\nu A^\rho$. This points to the Maxwell-Chern-Simons as being the dual model no matter what the gauge fixing term is.

3 Fields Redefinitions on Maxwell-Proca (MP) and its Dual

Another model which shares the same fields redefinition with its dual is Maxwell-Proca (MP),

$$S_{MP} = \frac{m^2}{2}(A, A) - \frac{1}{2}(*dA, *dA) = \frac{m^2}{2}(A, \left(1 - (*d/m)^2\right) A). \qquad (3.1)$$

We can find its dual following the gauge embedding procedure,

$$S_{dual-MP} = \frac{1}{2}(*dA, *dA) - \frac{1}{2m^2}(*d*dA, *d*dA). \qquad (3.2)$$

Just like SD model, MP can be redefined to a pure masslike term namely

$$S_{MP} = \frac{m^2}{2}(\hat{A}, \hat{A}) = \frac{m^2}{2}(OA, OA), \qquad (3.3)$$

where O also satisfies the properties (2.9). From (3.1) and (3.3), we see that O is nothing but:

$$O = (1 - (*d/m)^2)^{1/2}. \qquad (3.4)$$

Repeating the steps in section 2, we are left with:

$$[O^2 - 1 + (\frac{*d}{m})^2]A = 0. \qquad (3.5)$$

Applying $(*d)^2$ and integrating in A:

$$S_{dual-MP} = \frac{1}{2}(*dA, *dA) - \frac{1}{2m^2}(*d*dA, *d*dA) = \frac{1}{2}(*d\hat{A}, *d\hat{A}). \qquad (3.6)$$

Observe that applying $*d$ in (3.5) and integrating, one gets another theory with same field redefinition as MP that is not its dual. We clear this point in [6].

4 Conclusion

In this work we analysed the aspects of field redefinition and duality of gauge theories in three dimensions. We showed that the MCS and SD models can

be rewritten as a unique term with the same field redefinition. Indeed both models are members of a equivalence class of gauge theories. Let us remark that such redefinition is local and has a closed expression in operator $*d$, e.g, see equation (2.12). The expansion in power of $*d$ is formal and only makes sense only operating on a gauge field. Our results clarify some aspects of duality in three dimension. In this approach, the gauge fixing term drops out the dual theory.

Acknowledgments

A Functional calculus with differential forms

We start by defining the functional derivative of a p-form $A(x)$. Let the inner product of two p-form be given by

$$(A, B) = \int A(x) * B(x) = \int \frac{1}{p!} A(x)_{\mu_1..\mu_p} B(x)^{\mu_1..\mu_p} d^D x, \qquad (A.1)$$

where we are considering a flat manifold. Taking the functional derivative with respect to $A(y)_{\nu_1..\nu_p}$ on both sides of (A.1), one gets

$$\frac{\delta}{\delta A(y)_{\nu_1..\nu_p}}(A, B) = B(y)^{\nu_1..\nu_p}. \qquad (A.2)$$

Then we define the functional derivative of a p-form $A(x)$ as

$$\frac{\delta}{\delta A(y)}(A, B) = B(y). \qquad (A.3)$$

We also define the Dirac delta p-form

$$(\delta_p^D(x - y), B(x)) = B(y), \qquad (A.4)$$

then

$$\frac{\delta A(x)}{\delta A(y)} = \delta_p^D(x - y). \qquad (A.5)$$

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From definition (A.4), we can get the explicit form of $\delta_p^D(x - y)$ in terms of usual Dirac delta function:

$$\delta_p^D(x - y) = \frac{1}{p!}\delta^D(x - y)g_{\mu_1\nu_1}..g_{\mu_p\nu_p}dx^{\mu_1} \wedge .. \wedge dx^{\mu_p} \otimes dy^{\nu_1} \wedge .. \wedge dy^{\nu_p}. \, (\text{A}.6)$$

Clearly, $\delta_0^D(x - y) = \delta^D(x - y)$. Note that a p-form is defined in a point of the co-tangent space, therefore two p-forms in different points always commute since they are defined in different co-tangent spaces, i.e, $B(x)B(y) = B(y)B(x)$.

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